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A sufficient condition of validity for cubic Bézier triangles

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Abstract

The development of robust high-order finite element methods requires valid curvilinear discretizations for complex geometries without user intervention. The existing element validity verification methods are computationally and geometrically complicated. In this note, an efficient element validity verification procedure is developed for cubic Bézier triangles, and this verification can be extended to polynomials of arbitrary order Bézier elements.

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Keywords: validity; Jacobian; Bézier triangle; curvilinear mesh

1. Introduction

High-order finite element methods have been used extensively in direct numerical simulations in the last few decades. The high-order discretizations, which are required by the high-order finite element simulations, have been shown to offer exponential rates of convergence, small dispersion and diffusion solution errors [1,6]. Therefore, valid meshes with properly curved elements must be constructed to approximate the curved geometric domain.

When a geometric domain is given, the common way to accomplish the generation of a curvilinear mesh is to initially construct a straight-edge discretization of the model geometry, followed by the transformation of that discretization into high-order elements suitable for a high-order FE method. The invalid elements are usually caused by curving only the boundary mesh edges while the interior mesh edges remain straight. Thus, it is necessary to verify the validity and to eliminate all the invalid elements by curving interior mesh edges as a post-processing step once the curved mesh has been constructed.

When elements of the basic types are mapped into distorted forms, a general principle is that the mapping should admit an inverse, which means that it should be bijective. This implies that the sign of the *Jacobian* of the transformation has to be strictly positive everywhere on this element. Documents [5,7] have revealed that when the Lagrange form is selected to represent the curved meshes, it is unfeasible to decide the positiveness of the *Jacobian*. In order to obtain the lower bound on the *Jacobian*, one way is to adaptively expand the elementary *Jacobian* determinants (in Lagrange form) in Bézier basis that has properties of boundedness [3], the other way is to represent the curved

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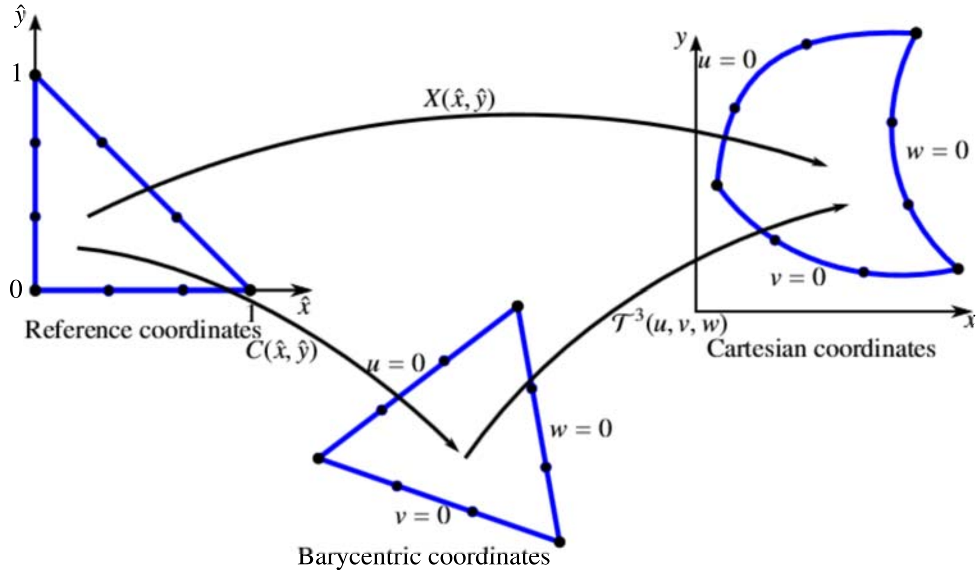


Fig. 1: Reference unit triangle in local coordinates $(\hat{x}, \hat{y})$ and the mappings $\mathbf{X}(\hat{x}, \hat{y})$, $\mathbf{C}(\hat{x}, \hat{y})$ and $\mathcal{T}(u, v, w)$. A general principle for the transformations: a one-to-one correspondence between coordinate systems.

elements directly using the Bézier form [4,7,8]. Taking advantage of the properties of Bézier element, the *Jacobian* expression can be formulated (it is a higher order Bézier triangle) and the lower bound can be calculated by the Bézier convex hull property [2]. To get the tight bound, the convex hull of the *Jacobian* is recursively refined using the Bézier subdivision algorithm [2]. However, even though the quadratic or cubic polynomials were selected, the evaluation is computationally and geometrically complicated. In this paper, an efficient element validity verification procedure is developed for cubic Bézier triangles, and this verification can be extended to polynomials of arbitrary order Bézier elements.

2. Element validation

Let's start with the definition of a Bézier triangle. The n-th order Bézier triangle is defined in terms of the barycentric coordinates $\mathbf{u} = (u, v, w)$ as follows:

$$\mathcal{T}^n(\mathbf{u}) = \sum_{|\mathbf{i}|=n} B^n_{\mathbf{i}}(\mathbf{u}) \mathcal{T}^0_{\mathbf{i}},$$

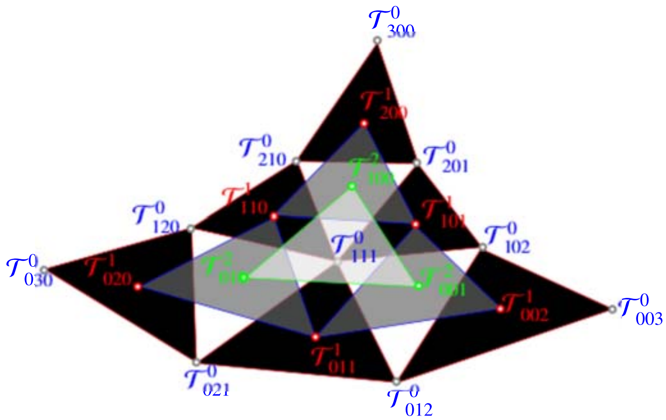
where

$$B^n_{\mathbf{i}}(\mathbf{u}) = \binom{n}{\mathbf{i}} u^i v^j w^k, \quad \mathbf{i} = (i, j, k), \quad |\mathbf{i}| = n, \quad \mathbf{u} = (u, v, w)$$

is the n-th order Bernstein polynomial, $u \in [0, 1]$, $v \in [0, 1]$, $w \in [0, 1]$ and $u + v + w = 1$. The set of points $\mathcal{T}^0_{\mathbf{i}}$ are *control points*, and the net formed by points $\mathcal{T}^0_{\mathbf{i}}$ is called *control net* of the Bézier triangle $\mathcal{T}^n$. Fig. 2 gives an example of a control net of a cubic triangle formed by its ten control points $\mathcal{T}^0_{\mathbf{i}}$. The reference triangle is first mapped to a triangle in barycentric coordinates (by the mapping $\mathbf{C}(\hat{x}, \hat{y})$) and then mapped to a curved triangle in global (x, y) coordinates (by the mapping $\mathcal{T}(u, v, w)$). This two-step mapping is presented in Fig. 1.

Jacobian is the determinant of the Jacobian matrix J which is defined by all first-order partial derivatives of the transformation:

$$J = \begin{bmatrix} \frac{\partial \mathcal{T}^n_x}{\partial \hat{x}} & \frac{\partial \mathcal{T}^n_x}{\partial \hat{y}} \\ \frac{\partial \mathcal{T}^n_y}{\partial \hat{x}} & \frac{\partial \mathcal{T}^n_y}{\partial \hat{y}} \end{bmatrix}.$$



$$
\begin{aligned}
\mathcal{T}^1_{200} &= u\mathcal{T}^0_{300} + v\mathcal{T}^0_{210} + w\mathcal{T}^0_{201},\\
\mathcal{T}^1_{110} &= u\mathcal{T}^0_{210} + v\mathcal{T}^0_{120} + w\mathcal{T}^0_{111},\\
\mathcal{T}^1_{101} &= u\mathcal{T}^0_{201} + v\mathcal{T}^0_{111} + w\mathcal{T}^0_{102},\\
\mathcal{T}^1_{020} &= u\mathcal{T}^0_{120} + v\mathcal{T}^0_{030} + w\mathcal{T}^0_{021},\\
\mathcal{T}^1_{011} &= u\mathcal{T}^0_{111} + v\mathcal{T}^0_{021} + w\mathcal{T}^0_{012},\\
\mathcal{T}^1_{002} &= u\mathcal{T}^0_{102} + v\mathcal{T}^0_{012} + w\mathcal{T}^0_{003},\\
\mathcal{T}^2_{100} &= u\mathcal{T}^1_{200} + v\mathcal{T}^1_{110} + w\mathcal{T}^1_{101},\\
\mathcal{T}^2_{010} &= u\mathcal{T}^1_{110} + v\mathcal{T}^1_{020} + w\mathcal{T}^1_{011},\\
\mathcal{T}^2_{001} &= u\mathcal{T}^1_{101} + v\mathcal{T}^1_{011} + w\mathcal{T}^1_{002},\\
\mathcal{T}^3_{000} &= u\mathcal{T}^2_{100} + v\mathcal{T}^2_{010} + w\mathcal{T}^2_{001}.
\end{aligned}
$$

Fig. 2: An illustration of the de Casteljau Algorithm for a cubic Bézier triangle. The control points $\mathcal{T}^0_{\mathbf{i}}$ form a control net, and the black triangles $\triangle\mathcal{T}^0_{300}\mathcal{T}^0_{210}\mathcal{T}^0_{201}$, $\triangle\mathcal{T}^0_{210}\mathcal{T}^0_{120}\mathcal{T}^0_{111}$, $\triangle\mathcal{T}^0_{201}\mathcal{T}^0_{111}\mathcal{T}^0_{102}$, $\triangle\mathcal{T}^0_{120}\mathcal{T}^0_{030}\mathcal{T}^0_{021}$, $\triangle\mathcal{T}^0_{111}\mathcal{T}^0_{021}\mathcal{T}^0_{012}$ and $\triangle\mathcal{T}^0_{102}\mathcal{T}^0_{012}\mathcal{T}^0_{003}$ are called control triangles.

Denote $A(\mathbf{p},\mathbf{q},\mathbf{r})$ the signed area of the triangle $\triangle\mathbf{pqr}$,

$$2A(\mathbf{p},\mathbf{q},\mathbf{r}) = \begin{vmatrix} 1 & 1 & 1 \\ p_x & q_x & r_x \\ p_y & q_y & r_y \end{vmatrix},$$

and $\mathbf{p} = (p_x, p_y)$, $\mathbf{q} = (q_x, q_y)$, $\mathbf{r} = (r_x, r_y)$. A triangle is considered not inverted if its vertices are labeled counterclockwise, meaning that the signed area of the triangle is positive.

Theorem 2.1. *A cubic Bézier triangle has strictly positive Jacobian if the control net of the cubic Bézier triangle is not twisted, meaning that all the control triangles (black triangles in Fig. 2 composed by control points) in the control net are not inverted.*

Proof. The proof mainly depends on the de Casteljau Algorithm for a cubic Bézier triangle [2] (illustrated in Fig. 2). Denote $\mathbf{e}1 = (1,0,0)$, $\mathbf{e}2 = (0,1,0)$, $\mathbf{e}3 = (0,0,1)$. Given a set of control points $\mathcal{T}^0_{\mathbf{i}} \in \mathcal{R}^2$ and barycentric coordinates $\mathbf{u} = (u, v, w)$, set

$$\mathcal{T}^r_{\mathbf{i}}(\mathbf{u}) = u\mathcal{T}^{r-1}_{\mathbf{i}+\mathbf{e}1}(\mathbf{u}) + v\mathcal{T}^{r-1}_{\mathbf{i}+\mathbf{e}2}(\mathbf{u}) + w\mathcal{T}^{r-1}_{\mathbf{i}+\mathbf{e}3}(\mathbf{u}), \quad r = 1,...,n, \quad |\mathbf{i}| = n - r,$$

then $\mathcal{T}^n_{\mathbf{0}}(\mathbf{u})$ is the point with parameter $\mathbf{u}$ on the n-th order Bézier triangle $\mathcal{T}^n$.

We first prove that the Jacobian has the same sign as the signed area of the triangle $\triangle\mathcal{T}^2_{100}\mathcal{T}^2_{010}\mathcal{T}^2_{001}$ in Fig. 2, then we observe that if there is no inverted control triangle, then the triangle $\triangle\mathcal{T}^2_{100}\mathcal{T}^2_{010}\mathcal{T}^2_{001}$ has positive area.

The Jacobian can be written as:

$$|J| = \begin{vmatrix} \frac{\partial\mathcal{T}^3_x}{\partial\hat{x}} & \frac{\partial\mathcal{T}^3_x}{\partial\hat{y}} \\ \frac{\partial\mathcal{T}^3_y}{\partial\hat{x}} & \frac{\partial\mathcal{T}^3_y}{\partial\hat{y}} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 3 & 0 & 0 \\ \frac{\partial\mathcal{T}^3_x}{\partial u} + \frac{\partial\mathcal{T}^3_x}{\partial v} + \frac{\partial\mathcal{T}^3_x}{\partial w} & \frac{\partial\mathcal{T}^3_x}{\partial\hat{x}} & \frac{\partial\mathcal{T}^3_x}{\partial\hat{y}} \\ \frac{\partial\mathcal{T}^3_y}{\partial u} + \frac{\partial\mathcal{T}^3_y}{\partial v} + \frac{\partial\mathcal{T}^3_y}{\partial w} & \frac{\partial\mathcal{T}^3_y}{\partial\hat{x}} & \frac{\partial\mathcal{T}^3_y}{\partial\hat{y}} \end{vmatrix}.$$

Because $u = 1 - \hat{x} - \hat{y}$, $v = \hat{x}$ and $w = \hat{y}$, we have $\frac{\partial u}{\partial\hat{x}} + \frac{\partial v}{\partial\hat{x}} + \frac{\partial w}{\partial\hat{x}} = 0$, $\frac{\partial u}{\partial\hat{y}} + \frac{\partial v}{\partial\hat{y}} + \frac{\partial w}{\partial\hat{y}} = 0$ and

$$\begin{vmatrix} 1 & \frac{\partial u}{\partial\hat{x}} & \frac{\partial u}{\partial\hat{y}} \\ 1 & \frac{\partial v}{\partial\hat{x}} & \frac{\partial v}{\partial\hat{y}} \\ 1 & \frac{\partial w}{\partial\hat{x}} & \frac{\partial w}{\partial\hat{y}} \end{vmatrix} = 3,$$

thus

$$|J| = \frac{1}{3}\begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial \mathcal{T}_x^3}{\partial u} & \frac{\partial \mathcal{T}_x^3}{\partial v} & \frac{\partial \mathcal{T}_x^3}{\partial w} \\ \frac{\partial \mathcal{T}_y^3}{\partial u} & \frac{\partial \mathcal{T}_y^3}{\partial v} & \frac{\partial \mathcal{T}_y^3}{\partial w} \end{vmatrix}\begin{vmatrix} 1 & \frac{\partial u}{\partial \hat{x}} & \frac{\partial u}{\partial \hat{y}} \\ 1 & \frac{\partial v}{\partial \hat{x}} & \frac{\partial v}{\partial \hat{y}} \\ 1 & \frac{\partial w}{\partial \hat{x}} & \frac{\partial w}{\partial \hat{y}} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial \mathcal{T}_x^3}{\partial u} & \frac{\partial \mathcal{T}_x^3}{\partial v} & \frac{\partial \mathcal{T}_x^3}{\partial w} \\ \frac{\partial \mathcal{T}_y^3}{\partial u} & \frac{\partial \mathcal{T}_y^3}{\partial v} & \frac{\partial \mathcal{T}_y^3}{\partial w} \end{vmatrix}.$$

Because $\frac{\partial \mathcal{T}_{200}^1}{\partial u}$, $\frac{\partial \mathcal{T}_{110}^1}{\partial u}$ and $\frac{\partial \mathcal{T}_{101}^1}{\partial u}$ can be computed as: (refer to Fig. 2)

$$\frac{\partial \mathcal{T}_{200}^1}{\partial u} = \frac{\partial(u\mathcal{T}_{300}^0 + v\mathcal{T}_{210}^0 + w\mathcal{T}_{201}^0)}{\partial u} = \mathcal{T}_{300}^0,$$

$$\frac{\partial \mathcal{T}_{110}^1}{\partial u} = \frac{\partial(u\mathcal{T}_{210}^0 + v\mathcal{T}_{120}^0 + w\mathcal{T}_{111}^0)}{\partial u} = \mathcal{T}_{210}^0,$$

$$\frac{\partial \mathcal{T}_{101}^1}{\partial u} = \frac{\partial(u\mathcal{T}_{201}^0 + v\mathcal{T}_{111}^0 + w\mathcal{T}_{102}^0)}{\partial u} = \mathcal{T}_{201}^0,$$

then

$$\begin{aligned}\frac{\partial \mathcal{T}_{100}^2}{\partial u} &= \frac{\partial(u\mathcal{T}_{200}^1 + v\mathcal{T}_{110}^1 + w\mathcal{T}_{101}^1)}{\partial u} \\ &= \mathcal{T}_{200}^1 + u\frac{\partial \mathcal{T}_{200}^1}{\partial u} + v\frac{\partial \mathcal{T}_{110}^1}{\partial u} + w\frac{\partial \mathcal{T}_{101}^1}{\partial u} \\ &= \mathcal{T}_{200}^1 + u\mathcal{T}_{300}^0 + v\mathcal{T}_{210}^0 + w\mathcal{T}_{201}^0 \\ &= \mathcal{T}_{200}^1 + \mathcal{T}_{200}^1 \\ &= 2\mathcal{T}_{200}^1.\end{aligned}$$

And $\frac{\partial \mathcal{T}_{010}^2}{\partial u}$, $\frac{\partial \mathcal{T}_{001}^2}{\partial u}$ can be derived similarly:

$$\frac{\partial \mathcal{T}_{010}^2}{\partial u} = 2\mathcal{T}_{110}^1,$$

$$\frac{\partial \mathcal{T}_{001}^2}{\partial u} = 2\mathcal{T}_{101}^1.$$

Thus,

$$\begin{aligned}\frac{\partial \mathcal{T}^3}{\partial u} &= \frac{\partial(u\mathcal{T}_{100}^2 + v\mathcal{T}_{010}^2 + w\mathcal{T}_{001}^2)}{\partial u} \\ &= \frac{\partial(u\mathcal{T}_{100}^2)}{\partial u} + \frac{\partial(v\mathcal{T}_{010}^2)}{\partial u} + \frac{\partial(w\mathcal{T}_{001}^2)}{\partial u} \\ &= \mathcal{T}_{100}^2 + u\frac{\partial \mathcal{T}_{100}^2}{\partial u} + v\frac{\partial \mathcal{T}_{010}^2}{\partial u} + w\frac{\partial \mathcal{T}_{001}^2}{\partial u} \\ &= \mathcal{T}_{100}^2 + u2\mathcal{T}_{200}^1 + v2\mathcal{T}_{110}^1 + w2\mathcal{T}_{101}^1 \\ &= 3\mathcal{T}_{100}^2.\end{aligned}$$

Similarly, we derive the following equations:

$$\frac{\partial \mathcal{T}^3}{\partial v} = 3\mathcal{T}_{010}^2,$$

$$\frac{\partial \mathcal{T}^3}{\partial w} = 3\mathcal{T}_{001}^2.$$

Therefore, the Jacobian can be computed as

$$
\begin{aligned}
|J| &= \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial \mathcal{T}_x^3}{\partial u} & \frac{\partial \mathcal{T}_x^3}{\partial v} & \frac{\partial \mathcal{T}_x^3}{\partial w} \\ \frac{\partial \mathcal{T}_y^3}{\partial u} & \frac{\partial \mathcal{T}_y^3}{\partial v} & \frac{\partial \mathcal{T}_y^3}{\partial w} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 1 \\ 3\mathcal{T}_{100x}^2 & 3\mathcal{T}_{010x}^2 & 3\mathcal{T}_{001x}^2 \\ 3\mathcal{T}_{100y}^2 & 3\mathcal{T}_{010y}^2 & 3\mathcal{T}_{001y}^2 \end{vmatrix} \\
&= 9 \begin{vmatrix} 1 & 1 & 1 \\ \mathcal{T}_{100x}^2 & \mathcal{T}_{010x}^2 & \mathcal{T}_{001x}^2 \\ \mathcal{T}_{100y}^2 & \mathcal{T}_{010y}^2 & \mathcal{T}_{001y}^2 \end{vmatrix} \\
&= 18A(\mathcal{T}_{100}^2, \mathcal{T}_{010}^2, \mathcal{T}_{001}^2).
\end{aligned}
$$

So the Jacobian has the same sign as the signed area of the triangle $\triangle \mathcal{T}_{100}^2 \mathcal{T}_{010}^2 \mathcal{T}_{001}^2$.

Because of the structure of the control net, when the vertices of all the triangles in the control net are in the counterclockwise order, the vertices of all the triangles in the second layer control net (formed by vertices $\mathcal{T}_{\mathbf{i}}^1$) are also in the counterclockwise order, so do the triangles in the third layer control net (formed by triangle $\triangle \mathcal{T}_{100}^2 \mathcal{T}_{010}^2 \mathcal{T}_{001}^2$). Thus, the intermediate triangle $\triangle \mathcal{T}_{100}^2 \mathcal{T}_{010}^2 \mathcal{T}_{001}^2$ has positive area, and therefore the Jacobian is positive. □

3. Conclusion

This paper provides a novel method for verifying the validity of curvilinear triangles represented by cubic Bézier polynomials and its proof. Since the run time of evaluating the sign of the signed area of a control triangle is constant, and for a fixed order of Bézier triangle, the number of control triangles in the control net is fixed, then the evaluation time for one Bézier triangle is constant. Compared to the previous methods that recursively split the convex hull of the Bézier triangle to get the lower bound of the Jacobian, this method is easy to implement and efficient. Bézier elements with polynomials of arbitrary order can be checked similarly. For the future work, we will evaluate this method using a variety of mesh examples of different orders.

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